Groups of homeomorphisms and diffeomorphisms of non-compact manifolds with the Whitney topology

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§1. Model spaces for Whitney Topology

Compact-Open Topology \leftrightarrow \text{Tychonoff Products / Weak Products of } l_2

Whitney Topology \leftrightarrow \text{Box Products / Small Box Products of } l_2

Definition. \quad \omega = \{0, 1, 2, \ldots\}

(1) The (Countable) Box product of Top space \( X \) : \( \Box^\omega X \)
- the usual countable product \( \prod_{n \in \omega} X \) as a Set
- is given Box Topology

This topology is generated by \( \prod_n U_n \) \((U_n \subset X : \text{Open subset})\)

(2) The Small box product of Pointed Space \((X, \ast)\) : \( \Box^\omega X \)
- Subspace of \( \Box^\omega X \) consisting of finite seq’s \((x_0, x_1, \ldots, x_i, \ast, \ast, \ldots)\)

(1) \( \Box^\omega l_2 \) : a bad space (not normal, not locally connected, etc.)

(2) Top. Classification of LF spaces \quad (P. Mankiewicz, 1974)
\[ \Box^\omega \mathbb{R} \approx \mathbb{R}^\infty \equiv \text{dir lim } \{\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots\} \]
\[ \Box^\omega l_2 \approx l_2 \times \mathbb{R}^\infty \]
§2. Homeomorphism Groups with Whitney topology

\( M \) : Non-compact \( n \)-manifold (possibly with boundary)

\( \mathcal{H}(M) \) : Homeo Group of \( M \) (Whitney topology) — Top group

\( * \quad \mathcal{O}(h, \mathcal{U}) := \{ g \in \mathcal{H}(M) \mid g: \mathcal{U}\text{-close to } h \} \quad (h \in \mathcal{H}(M), \ \mathcal{U} \in \text{cov}(M)) \)

\( \mathcal{H}_0(M) \) : the identity connected component of \( \mathcal{H}(M) \)

\( \mathcal{H}_c(M) \subset \mathcal{H}(M) \) : Subgroup of Homeo’s with compact support

Basic Property. (BMSY, 2011)

(1) \( \mathcal{H}_c(M) \) : Paracompact, Locally contractible

(2) \( \mathcal{H}_0(M) \subset \mathcal{H}_c(M) \) : Open normal subgroup

(3) \( \mathcal{M}_c(M) = \mathcal{H}_c(M)/\mathcal{H}_0(M) \) : Mapping Class Group (Discrete Top)

\[ \mathcal{H}_c(M) \cong \mathcal{H}_{0}(M) \times \mathcal{M}_c(M) \quad \text{(as Top spaces)} \]

(4) \( (M_i)_{i \in \mathbb{N}} \) : Compact subsets of \( M \) s.t. \( M_i \subset \text{Int}_M M_{i+1}, \ M = \bigcup_i M_i \)

\[ G(M_i) := \{ h \in \mathcal{H}_c(M) \mid \text{supp } h \subset M_i \} \]

\[ \mathcal{H}_c(M) = \varprojlim_i G(M_i) \quad \text{(Direct limit in Category of Top Groups)} \]
Topological Type

1-dim case \((\mathcal{H}(\mathbb{R}), \mathcal{H}_c(\mathbb{R})) \approx (\Box^\omega l_2, \Box^\omega l_2)\) (BMS, 2009)

2-dim case

\(M\) : Non-compact Connected 2-manifold (possibly with boundary)

Thm 1. \((\mathcal{H}(M), \mathcal{H}_c(M)) \approx_\ell (\Box^\omega l_2, \Box^\omega l_2)\) (BMS, 2011)

Thm 2. (1) \(\mathcal{H}_0(M) \approx \Box^\omega l_2 \approx l_2 \times \mathbb{R}^{\infty}\) (BMSY, 2014)

(2) \(\mathcal{H}_c(M) \approx \mathcal{H}_0(M) \times \mathcal{M}_c(M)\) \((\mathcal{M}_c(M) : \text{discrete})\)

\(\mathcal{M}_c(M) \approx \text{homeo} \left\{ \begin{array}{ll} \text{1pt} & M = X - K : (\ast) \smallskip \\
\text{N} & \text{in all other cases} \end{array} \right.\)

(\ast) : \(X = \text{Annulus, Disk or Möbius band,}\)

\(K = \text{Non-empty compact subset of one boundary circle of } X\)

Remark. \(\mathcal{H}(M)_{co}\) with Compact-Open Topology (T. Yagasaki, 2000)

\((\mathcal{H}(M)_{co})_0 \approx \left\{ \begin{array}{ll} S^1 \times l_2 & \text{if } M = \mathbb{R}^2, S^1 \times \mathbb{R}, S^1 \times [0, \infty) \text{ or } M - \partial M \\
l_2 & \text{in all other cases.} \end{array} \right.\)
§3. Diffeomorphism Groups with Whitney $C^\infty$-topology

$M$ : Non-compact $C^\infty$ $n$-manifold (without boundary)

$\mathcal{D}(M)$ : Diffeo Group of $M$ (Whitney $C^\infty$-topology) — Top group

$\mathcal{D}_0(M)$ : the identity connected component of $\mathcal{D}(M)$

$\mathcal{D}_c(M) \subset \mathcal{D}(M)$ : Subgroup of Diffeo’s with compact support

Basic Property. (BMSY, 2011)

(1) $\mathcal{D}_c(M)$ : Paracompact

(2) $\mathcal{D}_0(M) \subset \mathcal{D}_c(M)$ : Open normal subgroup

(3) $\mathcal{M}_c^\infty(M) = \mathcal{D}_c(M)/\mathcal{D}_0(M)$ : Mapping Class Group (Discrete Top)

\[ \mathcal{D}_c(M) \cong \mathcal{D}_0(M) \times \mathcal{M}_c^\infty(M) \] (as Top spaces)

homeo

(4) $(M_i)_{i \in \mathbb{N}}$ : Compact subsets of $M$ s.t. $M_i \subset \text{Int}_MM_{i+1}$, $M = \bigcup_i M_i$

$G(M_i) := \{ h \in \mathcal{D}_c(M) \mid \text{supp } h \subset M_i \}$

$\mathcal{D}_c(M) = \underset{i}{\text{g-lim}} \ G(M_i)$ (Direct limit in Category of Top Groups)
Topological Type

1-dim case \((\mathcal{D}(\mathbb{R}), \mathcal{D}_c(\mathbb{R})) \approx (\Box^{\omega}_2, \Box^{\omega}_2)\) (BY, 2010)

n-dim case

\(M\) : Non-compact Connected \(C^\infty\) \(n\)-manifold (without boundary)

**Thm 1.** \((\mathcal{D}(M), \mathcal{D}_c(M)) \approx_\ell (\Box^{\omega}_2, \Box^{\omega}_2)\) (BMSY, 2011)

**Thm 2.** \(\mathcal{D}_c(M) \approx\) Open subset of \(l_2 \times \mathbb{R}^\infty\) (BY)

\(\mathcal{D}_0(M) \approx L \times \mathbb{R}^\infty\) for some top. \(l_2\)-manifold \(L\) \(\ (L \simeq \mathcal{D}_0(M))\)

(1) \(\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty\) if \(n = 1, 2\) or

\(n = 3\) and \(M\) : orientable, irreducible

(2) \(\mathcal{D}_0(M) \approx \mathcal{D}_0(N; \partial N) \times \mathbb{R}^\infty\)

if \(N\) : Compact Connected \(C^\infty\) \(n\)-manifold with Boundary

\(M = \text{Int } N\)
§5. Criterion of Top Group $\approx l_2 \times \mathbb{R}^\infty$.

T. Banakh - D. Repovš (2009 - ) — Series of papers

Study of Direct limit of Uniform spaces and Top. LF-manifolds
Quotients of Hilbert manifolds groups, etc.

Thm. (BMRSY, 2013)

$G :$ Top Group, Non-metrizable, $G = \cup_n G_n$

(*1) $G_n \subset G :$ Closed subgroup, $G_n \subset G_{n+1}, \ G_n \approx l_2$

(*2) $p : \sqcap_n G_n \to G :$ Open

(*3) $G_{n+1} \to G_{n+1}/G_n \text{ admits a local section}$

(*4) each $Z$-point of $G_{n+1}/G_n$ is a strong $Z$-point.

$\implies G \approx l_2 \times \mathbb{R}^\infty$
References.

BMRSY = T. Banakh, K. Mine, D. Repovš, K. Sakai, T. Yagasaki


Thank you very much for your attention!