Local and End Deformation Properties for Uniform Embeddings

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The 8th Waseda Geometric Topology Meeting
Waseda Univ., Sep. 5, 2015
§1. Introduction

§1.1. Basic Definitions - I

\( f : (X, d) \rightarrow (Y, \varrho) \)  \hspace{1cm} (\( X, d \), \( Y, \varrho \)) : Metric spaces

(1) \( f \) : Uniformly continuous  \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } x, x' \in X, \ d(x, x') < \delta \implies \varrho(f(x), f(x')) < \varepsilon \\

(2) \( f \) : Uniform homeomorphism  \iff f : \text{bijective} \quad f, f^{-1} : \text{uniformly continuous} \\

(3) Uniform embedding = Uniform homeomorphism onto its image

\( f, g : (X, d) \rightarrow (Y, \varrho) \)

(4) Sup-metric :  \( \varrho(f, g) = \sup \{ \varrho(f(x), g(x)) \mid x \in X \} \in [0, \infty] \)

(5) Uniform Topology \iff Sup-metric

Metric Manifold = Top Manifold with Fixed metric
§1.2. Problem

\((X, d) :\) Metric space \(A \subset X\)

\(\mathcal{H}^u(X; A) =\) Group of uniform homeomorphisms

\(h : X \approx X\) \quad s.t. \quad h = \text{id} \text{ on } A

(Sup-metric \& Uniform Topology)

**Goal** \((M, d) :\) Non-Compact Metric Manifold

Understand (Local and Global) Topological Properties of \(\mathcal{H}^u(M, d)\)

\(\circ \ \mathcal{H}^u(M, d) :\) Local Contractibility \& Contractibility
Historical Remarks. In study of groups of homeomorphisms, PL-homeomorphisms, diffeomorphisms etc.

(1) Direct Study of topological properties as topological spaces is **not** a standard approach to these groups even for Compact manifolds.

(2) A standard approach is to study some approximations of these groups

   (i) The approximations are constructed by “simplices” which respect the structures TOP, PL, Diff, etc.

   (ii) This standard approach has many important applications.

   (iii) Study of relations between these approximations and typical topologies on the groups is another problem.

     - For the groups of PL-homeomorphisms, there is no typical topology which respects the PL-structure.
(3) For Non-compact manifolds,

these groups does not have enough good topological properties

under the typical topologies $\tau = \begin{cases} 
\text{Compact-open topology} \\
\text{Uniform topology} \\
\text{Whitney topology}
\end{cases}$

$\mathcal{H}(M)_\tau$ is not necessarily locally contractible as a topological space

To obtain some affirmative results, people adopted weaker notions:

In the standard textbook “Topological Embeddings” (T. B. Rushing)

a weaker notion of local contractibility & contractibility is introduced.

(i) (Local) Contractibility of $\mathcal{H}(M)_\tau$

$= (\text{Local})$ Existence of continuous selections of isotopies to $\text{id}_M$

under the topology $\tau$.

(ii) Weaker than the usual (Local) Contractibility as topological spaces.

(iii) In this weaker sense

$\mathcal{H}(M)_w$ is locally contractible (R. D. Edwards & R. C. Kirby (1971))

$\mathcal{H}_b^u(\mathbb{R}^n)_u$ is contractible by Alexander trick (J. M. Kister (1969))
§2. Preliminaries

§2.1. Subgroups of $\mathcal{H}^u(X)$

$(X, d) : \text{Metric space} \quad A \subset X$

$\mathcal{H}^u(X; A) = \text{Group of uniform homeomorphisms} \\
h : X \approx X \quad \text{s.t.} \quad h = \text{id on } A$

(Sup-metric & Uniform Topology)

Subgroups of $\mathcal{H}^u(X; A)$ :

$\mathcal{H}^u(X; A) \triangleright \mathcal{H}^u_b(X; A) \triangleright \mathcal{H}^u(X; A)_0$

open & closed

$\mathcal{H}^u_b(X; A) \triangleright \mathcal{H}_0(X; A) \triangleright \mathcal{H}_c(X; A)$

(1) $\mathcal{H}^u_b(X; A) : \text{Subgroup of Bounded Homeo’s} \quad d(h, \text{id}_X) < \infty$

(2) $\mathcal{H}^u(X; A)_0 : \text{Connected component of } \text{id}_X$

(3) $\mathcal{H}_0(X; A) : \text{Subgroup of Homeo’s Asymptotic to } \text{id}_X$

\[ (\forall \varepsilon > 0 \exists K \subset X : \text{Compact \ s.t.} \quad d(h(x), x) < \varepsilon \ (x \in X - K) ) \]

(4) $\mathcal{H}_c(X; A) : \text{Subgroup of Homeo’s with Compact Support}
\section*{2.2. $\kappa$-cones \hspace{0.5em} ($\kappa \leq 0$)}

$(Y, d)$ : a metric space

- the cone : \[ C(Y) := (Y \times [0, \infty)) / (Y \times \{0\}) \quad ty = [y, t] \]
- the cone ends : \[ C(Y)_r := \{ty \in C(Y) \mid t \geq r\} \quad (r > 0) \]
- the metric $d_\pi$ on $Y$ : \[ d_\pi(x, y) := \min\{d(x, y), \pi\} \quad (x, y \in Y) \]

**Definition**

(1) The $\kappa$-cone $C_\kappa(Y, d)$ over $(Y, d)$

\[ = \text{the metric space } (C(Y), \tilde{d}_\pi) : \]

(i) for $\kappa = 0$ : \[ \tilde{d}_\pi(sx, ty)^2 = s^2 + t^2 - 2st \cos d_\pi(x, y) \]

(ii) for $\kappa < 0$ :

\[ \cosh(\sqrt{-\kappa} \tilde{d}_\pi(sx, ty)) = \cosh(\sqrt{-\kappa} s) \cosh(\sqrt{-\kappa} t) \]

\[ - \sinh(\sqrt{-\kappa} s) \sinh(\sqrt{-\kappa} t) \cos d_\pi(x, y) \]

(2) The $\kappa$-cone ends $C_\kappa(Y, d)_r$ ($r > 0$) over $(Y, d)$

\[ = \text{the submetric space } (C(Y)_r, \tilde{d}_\kappa) \subset C_\kappa(Y, d) \]
Example. (The $n$-dim space form $M^n_\kappa$ of the sectional curvature $\kappa$)

$S^{n-1} = \{ \mathbf{x} \in \mathbb{R}^n \mid \| \mathbf{x} \| = 1 \}$: the unit $(n - 1)$-sphere in $\mathbb{R}^n$

(the standard spherical metric $d_1$)

$C_\kappa(S^{n-1})$ is isometric to the space form $M^n_\kappa$.

Remark. ($\kappa \leq 0$)

(1) The function $\lambda_\kappa : \mathbb{R} \to \mathbb{R}$:

$\lambda_\kappa(u) = \begin{cases} \frac{u}{2} & (\kappa = 0) \\ \sinh \left( \frac{\sqrt{-\kappa}}{2} u \right) & (\kappa < 0) \end{cases}$

(i) $\lambda_\kappa$: a diffeomorphism, a monotonically increasing odd function

(ii) $\lambda^2_\kappa(u) = (\lambda_\kappa(u))^2$.

(2) $\lambda^2_\kappa(\tilde{d}_\pi(sx, ty)) = \lambda^2_\kappa(s - t) + \lambda_\kappa(2s)\lambda_\kappa(2t) \sin^2 \frac{1}{2}d_\pi(x, y)$ ($sx, ty \in C(Y)$)

(i) $\tilde{d}_\pi(sy, ty) = |s - t| \leq \tilde{d}_\kappa(sx, ty)$

(ii) $\lambda_\kappa(\tilde{d}_\pi(tx, ty)) = \lambda_\kappa(2t) \sin \frac{1}{2}d_\pi(x, y)$. 
§3. Uniform isotopies vs Level-wise uniform isotopies

§3.1. Uniform isotopies

$(X, d) :$ a metric space

$$(X \times [0, 1], \tilde{d}) \quad \tilde{d}((x, t), (y, s)) = d(x, y) + |t - s|$$

$\quad ((x, t), (y, s) \in X \times [0, 1])$

**Definition.**

(1) An isotopy on $X$ is a homeomorphism $H : X \times [0, 1] \longrightarrow X \times [0, 1]$ which preserves $[0, 1]$-factor

(i) $H(x, t) = (H_t(x), t) \quad ((x, t) \in X \times [0, 1])$

(ii) $H_t \in \mathcal{H}(X) \quad (t \in [0, 1])$

(2) An isotopy $H$ on $X$ is said to be

(i) a (bounded) **uniform isotopy** on $(X, d)$ if $H \in \mathcal{H}_{(b)}^u(X \times [0, 1], \tilde{d})$

(ii) a level-wise (bounded) uniform isotopy on $(X, d)$

if $H_t \in \mathcal{H}_{(b)}^u(X, d) \quad$ for each $t \in [0, 1]$. 
Lemma.

\[ H = (H_t) : \text{a uniform isotopy} \quad \iff \quad [0, 1] \xrightarrow{t} \mathcal{H}^u(X, d) : \text{continuous} \]

(i) Uniform isotopies on \((X, d)\) \iff Conti. paths in \(\mathcal{H}^u(X, d)\)

(ii) \(\mathcal{H}^u_{(b)}(X \times [0, 1], \tilde{d})^I, \tilde{d}) \xrightarrow{\eta} (\mathcal{C}([0, 1], (\mathcal{H}^u_{(b)}(X, d), d))_u, d)\)

\[ H = (H_t) \quad \text{an isometry} \quad \widehat{H} : t \longmapsto H_t \]

Proposition. (Exponential Law) \(Z\) : any topological space

\(\mathcal{C}(Z \times [0, 1], \mathcal{H}^u_{(b)}(X, d))\) \(\varphi\)

\(\chi \downarrow \cong \text{an isometry}\)

\(\mathcal{C}(Z, \mathcal{C}([0, 1], \mathcal{H}^u_{(b)}(X, d))_u)\) \(\psi = \chi(\varphi) = \eta\Phi\)

\(\eta \# \uparrow \cong \text{an isometry}\)

\(\mathcal{C}(Z, \mathcal{H}^u_{(b)}(X \times [0, 1], \tilde{d})^I)\) \(\Phi\)

\(\varphi(z, t) = \psi(z)(t) = \Phi(z)_t \quad ((z, t) \in Z \times [0, 1])\)
§3.2. Alexander trick in $\kappa$-cones ($\kappa \leq 0$)

$(X, d) : a compact metric space \quad C_\kappa(X, d) = (C(X), \tilde{d}_\kappa)$

$$\theta_t \in \mathcal{H}(C_\kappa(X, d)) : \quad \theta_t(sx) = (ts)x \quad (t \in (0, \infty))$$

**Definition.** \quad $h \in \mathcal{H}_b(C_\kappa(X, d))$

(1) $h_t \in \mathcal{H}_b(C_\kappa(X, d)) \quad (t \in [0, 1]) : \quad h_t = \begin{cases} \theta_t h(\theta_{1/t}) & (t \in (0, 1]), \\ \text{id} & (t = 0). \end{cases}$

(2) (Alexander isotopy for $h$)

$$\Phi(h) : C_\kappa(X, d) \times [0, 1] \to C_\kappa(X, d) \times [0, 1] : \quad \Phi(h)(u, t) = (h_t(u), t)$$

**Basic Properties**

(1) $\Phi(h)$ is a level-wise bounded isotopy from $\text{id}_X$ to $h$.

(2) $\Phi : \mathcal{H}_b(C_\kappa(X, d)) \longrightarrow \mathcal{H}_{lb}(C_\kappa(X, d) \times [0, 1], \tilde{d}_\kappa)I$

$$h \quad \mapsto \quad \Phi(h)$$

(i) an isometric embedding

(ii) a contraction of $\mathcal{H}_b(C_\kappa(X, d))$ in the weak sense (of Rushing).
(3) $\eta_h : [0, 1] \rightarrow \mathcal{H}(C_\kappa(X, d)) : \eta_h(t) = h_t$ : a function

(i) $\eta_h$ is continuous at $t = 0$

(ii) $\Phi(h)$ : a uniform isotopy $\iff \eta_h$ is continuous

$$h_t \in \mathcal{H}_b^u(C_\kappa(X, d)) \ (t \in [0, 1]).$$

(iii) $\Phi(h)$ is not necessarily a uniform isotopy.

$\circ \exists h \in \mathcal{H}_b^u(\mathbb{R}^n)$ s.t. $\eta_h$ is not continuous.

(4) $\varphi : \mathcal{H}_b(C_\kappa(X, d)) \times [0, 1] \rightarrow \mathcal{H}_b(C_\kappa(X, d)) : \varphi(h, t) = h_t$

(i) $\varphi$ is not necessarily continuous on $\mathcal{H}_b^u(C_\kappa(X, d)) \times [0, 1]$.

(ii) $\mathcal{G} := \{ h \in \mathcal{H}_b^u(C_\kappa(X, d)) \mid \Phi(h)$ is Uniform isotopy $\} < \mathcal{H}_b^u(C_\kappa(X, d))$

(a) $\varphi$ is continuous on $\mathcal{G} \times [0, 1]$

(b) $\varphi$ induces contractions of the following subgroups :

$$\mathcal{G} > \mathcal{H}_0(C_\kappa(X, d)) > \mathcal{H}_c(C_\kappa(X, d))$$
Example.

\( \forall v \in \mathbb{R}^n - \{0\} \) we can find \( h \in \mathcal{H}_b^u(\mathbb{R}^n) \) s.t.

(i) \( h((2k + 1)v) = (2k + 1)v \) \( (k \in \mathbb{N}) \),

(ii) \( \exists c > 0 \) s.t. \( d(h(2kv), 2kv) > c \) \( (k \in \mathbb{N}) \).

For any such \( h \), the function \( \eta_h \) is not continuous at \( t = 1 \).

Proof. Assume that \( \eta_h \) is continuous at \( t = 1 \).

\( \exists t_0 \in [0, 1) \) s.t. \( d(h_t, h) < c \) \( (t \in (t_0, 1]) \).

\( \frac{2k}{2k+1} \to 1 \) \( (k \to \infty) \) \( \therefore \exists k \in \mathbb{N} \) s.t. \( t := \frac{2k}{2k+1} > t_0 \).

\( \therefore h_t(2kv) = th \left( \frac{1}{t} 2kv \right) = th((2k + 1)v) = t(2k + 1)v = 2kv \)

\( \therefore c > d(h_t, h) \geq d(h_t(2kv), h(2kv)) = d(2kv, h(2kv)) > c. \)  

(a contradiction !)
Further Notations.

(1) \((X, d)\) : a metric space \(A \subset B \subset X\)

\[ A \subset_u B \text{ in } (X, d) \iff O_\varepsilon(A) \subset B \text{ for some } \varepsilon > 0 \]

\((B : \text{Uniform neighborhood of } A \text{ in } X)\)

(2) \((M, d)\) : Metric \(n\)-manifold \(X, C \subset M\)

\(\mathcal{E}^u_*(X, M; C)\) : Space of uniform proper embeddings

\[ f : X \to M \quad \text{s.t.} \quad f = \text{id on } X \cap C \]

(Sup-metric & Uniform Topology)

\(\circ ~ i_X : X \subset M : \text{the inclusion map}\)

(i) \(\mathcal{E}^u_*(X, M; C)_b = \{ f \in \mathcal{E}^u_*(X, M; C) \mid d(f, i_X) < \infty \}\)

(ii) \(\mathcal{E}^u_*(i_X, \varepsilon; X, M; C) = \text{the open } \varepsilon\text{-neighborhood of } i_X \text{ in } \mathcal{E}^u_*(X, M; C)\)

\(\mathcal{H}^u(M)\)

(i) \(\mathcal{H}^u(\text{id}_M, \varepsilon; M) := \text{the open } \varepsilon\text{-neighborhood of } \text{id}_M \text{ in } \mathcal{H}^u(M)\)
§4. Local Deformation Property of Uniform embeddings (LD)

§4.1. Definition \((M, d)\) : Metric \(n\)-manifold

Definition. \(M : (LD)\)

\(\iff \ \forall (X, W', W, Z, Y)\) s.t. \(X \subset_u W' \subset W \subset M, \ Z \subset_u Y \subset M\)

\(\exists\ \) Neighborhood \(\mathcal{W}\) of \(i_W : W \subset M\) in \(\mathcal{E}^u(W, M; Y)\)

\(\exists\ \) Homotopy \(\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u(W, M; Z)\) s.t.

1. \(\forall f \in \mathcal{W}\)
   (i) \(\varphi_0(f) = f\)  \(\quad\) (ii) \(\varphi_1(f) = \text{id}\) on \(X\)
   (iii) \(\varphi_t(f) = f\) on \(W - W'\) and \(\varphi_t(f)(W) = f(W)\) \((t \in [0, 1])\)
   (iv) if \(f = \text{id}\) on \(W \cap \partial M\) then \(\varphi_t(f) = \text{id}\) on \(W \cap \partial M\) \((t \in [0, 1])\)

2. \(\varphi_t(i_W) = i_W\) \((t \in [0, 1])\)

Uniform Version of Local Deformation Theorem for

Topological Embeddings of Compact subspaces in Manifolds

(R. D. Edwards & R. C. Kirby (1971))
Criterion of (LD) — Decomposition of $M$ into Simpler Pieces

○ Need to Formulate (LD) for Subsets of $M$

Definition. $A \subset M$

$A : (\text{LD})_M \iff$

$\forall (X, W', W, Z, Y) \ s.t. \ X \subset A, \ X \subset_u W' \subset W \subset M, \ Z \subset_u Y \subset M$

$\exists$ Neighborhood $\mathcal{W}$ of $i_W : W \subset M$ in $\mathcal{E}_{u}^*(W, M; Y)$

$\exists$ Homotopy $\varphi : \mathcal{W} \times [0, 1] \rightarrow \mathcal{E}_{u}^*(W, M; Z)$ s.t.

(1) $\forall f \in \mathcal{W}$

(i) $\varphi_0(f) = f$    (ii) $\varphi_1(f) = \text{id}$ on $X$

(iii) $\varphi_t(f) = f$ on $W - W'$ and $\varphi_t(f)(W) = f(W)$ $(t \in [0, 1])$

(iv) if $f = \text{id}$ on $W \cap \partial M$ then $\varphi_t(f) = \text{id}$ on $W \cap \partial M$ $(t \in [0, 1])$

(2) $\varphi_t(i_W) = i_W$ $(t \in [0, 1])$

Remark. $M : (\text{LD}) \iff M : (\text{LD})_M$
§4.2. Formal Properties of (LD)

(1) (Invariance under Uniform Homeo’s)

\[ h: M \cong N \] (uniform homeo), \( A : (LD)_M \implies h(A) : (LD)_N \)

(2) (Restriction) \( A \subset B \subset M, \ B : (LD)_M \implies A : (LD)_M \)

(3) (Additivity) \( A \subset_u U \subset M, \ B \subset M \)

\[ U, B : (LD)_M \implies A \cup B : (LD)_M \]

(4) (Relatively compact subsets) \( K \subset M : \) Relatively compact

(i) \( K : (LD)_M \) (Edwards - Kirby)

(ii) \( A \subset M \)

\[ A : (LD)_M \iff A \cup K : (LD)_M \]

(5) (Neighborhoods of Ends) \( M = K \cup \bigcup_{i=1}^{m} L_i \)

\( K \subset M : \) Relatively compact

\( L_i \subset M : \) Closed, \( L_i : n\)-manifold, \( d(L_i, L_j) > 0 \) \( (i \neq j) \)

\( M : (LD) \iff L_i : (LD) \) \( (i = 1, \cdots, m) \)

(6) \( M : (LD) \implies \mathcal{H}^u(M) : \) Locally contractible
§4.3. Examples

$(M, d) : $ Metric manifold

[1] Metric covering spaces over compact manifolds

$\pi : (M, d) \to (N, \varrho) : $ Metric covering projection $\implies (M, d) : (LD)$

$N : $ Compact manifold

[2] Metric manifolds with geometric group actions

$(M, d) \text{ admits Geometric group action} \implies (M, d) : (LD)$

[3] $\kappa$-cones over Compact Lipschitz metric manifolds $(\kappa \leq 0)$

$(N, d) : $ Compact Lipschitz metric manifold $\implies C_\kappa(N, d)_1 : (LD)$

(i) Lipschitz metric $n$-manifold $=$ Metric $n$-manifold $(M, d)$ s.t.

$\forall \ x \in M \ \exists \ Open \ neighborhood \ U \ of \ x \ in \ M$ 

$\exists \ Open \ subset \ V \ of \ (\mathbb{R}^n_{\geq 0}, d_0) \ s.t.$

$(U, d|_U)$ is bi-Lipschitz homeomorphic to $(V, d_0|_V)$. 

$\circ$ Riemannian manifolds
Proof. \((\mathbb{S}^{n-1}, d_1)\)-case \(\Rightarrow\) General case

\[
\uparrow
\]

Additivity of (LD)

Invariance under Uniform Homeo’s

\((\mathbb{S}^{n-1}, d_1)\)-case:\hfill \(C_\kappa(\mathbb{S}^{n-1}, d_1) \approx M^n_\kappa\) : (LD)

(1) \(M^n_0 = \mathbb{R}^n\) (Euclidean \(n\)-space)

\(\mathbb{Z}^n \curvearrowright \mathbb{R}^n\) : Canonical geometric group action

\(\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n\) : Riem. covering projection onto Flat torus

(2) \(M^n_{-1} = \mathbb{H}^n\) (Hyperbolic \(n\)-space)

\(\exists \pi : \mathbb{H}^n \to L^n\) : Riem. covering projection

onto Compact Hyperbolic manifold

(3) \(\kappa < 0\) \(M^n_\kappa \cong \mathbb{H}^n\) (homothetic)

General case:

\((\mathbb{S}^{n-1}, d_1)\)-case + (LD) : Additivity

Invariance under Uniform Homeo’s
§5. End deformation property for Uniform embeddings (ED)

§5.1. Proper product ends

Definition.

(1) $n$-dim proper product end = a metric $n$-manifold $(L,d)$ s.t.

(i) $d$ is proper  
(ii) $\exists$ Compact $(n - 1)$-manifold $S$

$\exists$ Homeomorphism $\theta : S \times [1, \infty) \approx L$

(2) $(L,d)$ : Proper product end  

(Fix $\theta : S \times [1, \infty) \approx L$)

(i) $L_r := \theta(S \times [r, \infty))$ ($r \geq 1$)

(ii) $F \subseteq L :$ cofinal $\iff L_r \subseteq F$ for some $r \in [1, \infty)$

(3) $(M,d)$ : Metric $n$-manifold

(i) A proper product end of $(M,d)$

$= a$ closed subset $L$ of $M$ s.t. $(L,d|_L)$ is a proper product end.

Fr$_M L = \theta(S \times \{1\})$ (compact)

(ii) $L :$ Proper product end of $M$

$L$ is isolated $\iff d(M - L, L_r) \to \infty$ as $r \to \infty$
§5.2. End deformation property for Uniform embeddings (ED)

\((L, d) : \) a proper product end

**Definition.** \((L, d) : (ED) \iff\)

\(\forall F \subset L : \) conformal \quad \forall \alpha > 0

\(\exists H \subset F : \) conformal \quad \exists \beta > 0

\(\exists \varphi : \mathcal{E}^u_*(i_F, \alpha; F, L) \times [0, 1] \longrightarrow \mathcal{E}^u_*(i_F, \beta; F, L) \) s.t.

\((1) \forall f \in \mathcal{E}^u_*(i_F, \alpha; F, L)

\(\varphi_0(f) = f \quad (\text{ii}) \varphi_1(f) = \text{id } \text{on } H\)

\(\varphi_t(f) = f \quad \text{on } \text{Fr}_LF \quad (\text{and } \varphi_t(f)(F) = f(F)) \quad (t \in [0, 1])\)

\(\varphi_t(f) = \text{id } \text{on } F \cap \partial L \quad (t \in [0, 1]).\)

\((2) \varphi_t(i_F) = i_F \quad (t \in [0, 1])\)
Basic Properties of (ED).

(1) \((L, d), (L', d')\) : Proper product ends

\(\exists h : (L, d) \approx (L', d')\) : Uniform, coarsely uniform homeomorphism

\((L, d) : (ED) \implies (L', d') : (ED)\)

(2) (Example) \((N, d)\) : Compact metric manifold

\(C_0(N, d)_1 : (LD) \implies C_0(N, d)_1 : (ED)\)

(i) \((N, d)\) : Compact Lipschitz metric manifold \(\implies C_0(N, d)_1 : (ED)\)
§5.3. End deformation theorem for Uniform embeddings

\((M, d) : \text{Metric } n\text{-manifold}\)

\(L(1), \cdots, L(m) \subset M : \text{Disjoint isolated proper product ends of } M\)

\(\text{s.t. } (L(i), d|_{L(i)}) : (\text{ED}) \ (i = 1, \cdots, m)\)

**Theorem.**

\(\forall F \subset M : \text{closed s.t. } F \cap L(i) \subset L(i) : \text{cofinal } (i = 1, \cdots, m)\)

\(\forall s_i > r_i > 1 \text{ s.t. } L(i)_{r_i} \subset \text{Int}_M(F \cap L(i)) \ (i = 1, \cdots, m)\)

\(\exists \text{ a strong deformation retraction} \)

\[\varphi_t : \mathcal{E}_u^*(F, M)_b \searrow \mathcal{E}_u^*(F, M; \bigcup_{i=1}^m L(i)_{s_i})_b \text{ s.t.} \]

1. \(\forall f \in \mathcal{E}_u^*(F, M)_b \quad \forall t \in [0, 1] \)
   
   (i) \(\varphi_t(f) = f \text{ on } f^{-1}(M - \text{Int}_M(\bigcup_{i=1}^m L(i)_{r_i})) - \text{Int}_M(\bigcup_{i=1}^m L(i)_{r_i})\),
   
   (ii) if \(\bigcup_{i=1}^m L(i)_{r_i} \subset f(F)\) then \(\varphi_t(f)(F) = f(F)\),
   
   (iii) if \(f = \text{id} \text{ on } F \cap \partial M\) then \(\varphi_t(f) = \text{id} \text{ on } F \cap \partial M\).

2. \(\forall \alpha > 0 \ \exists \beta > 0 \text{ s.t.} \)

\[\varphi_t(\mathcal{E}_u^*(i_F, \alpha; F, M) \subset \mathcal{E}_u^*(i_F, \beta; F, M) \quad (t \in [0, 1])\]
\[ L_r := L(1)_r \cup \cdots \cup L(m)_r \quad (r \geq 1) \]

**Corollary.**

\[ \exists \text{ a strong deformation retraction } \varphi_t : \mathcal{H}^u_b(M) \searrow \mathcal{H}^u_b(M; L_3) \quad \text{s.t.} \]

1. \( \forall h \in \mathcal{H}^u_b(M) \quad \forall t \in [0, 1] \)
   
   (i) \( \varphi_t(h) = h \) on \( h^{-1}(M - \text{Int}_M L_2) - \text{Int}_M L_2 \)

   (ii) if \( h = \text{id} \) on \( \partial M \) then \( \varphi_t(h) = \text{id} \) on \( \partial M \).

2. \( \forall \alpha > 0 \quad \exists \beta > 0 \quad \text{s.t.} \)

\[ \varphi_t(\mathcal{H}^u(\text{id}_M, \alpha; M)) \subset \mathcal{H}^u(\text{id}_M, \beta; M) \quad (t \in [0, 1]) \]

**Example.** \( \mathcal{H}^u_b(\mathbb{R}^n) \simeq * \)

\( \therefore \) \( \mathcal{H}^u_b(\mathbb{R}^n) \searrow \mathcal{H}(\mathbb{R}^n; \mathbb{R}^n_3) \simeq \mathcal{H}(\mathbb{D}^n(3), \partial) \simeq * \)
End of Talk

Thank you very much
for your attention!