The uniform perfectness of diffeomorphism groups
of open manifolds

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§1. Background.

Algebraic property of diffeomorphism groups

Perfectness & Simplicity: (1970’s)

[1] M. Herman, W. Thurston, J. Mather, D. B. A. Epstein:

\( M : \) a \( \sigma \)-compact (separable metrizable) \( C^\infty \) \( n \)-manifold without boundary

\( \implies \) Diff\(_r^c(M)\)\(_0\) : perfect

simple if \( M \) is connected.


\( M = \text{Int} \, W \) (\( W \) : a compact manifold with \( \partial \neq \emptyset \))

\( \implies \) Diff\(_r(M)\)\(_0\) : perfect

Uniform Perfectness, Boundedness, Uniform Simplicity: (2000’s)

Conjugation-invariant norms on diffeomorphism groups

Commutator length, Conjugation-generated norm


cld \text{Diff}^r(S^n)_0 \leq 4, \quad M: \text{a closed 3-manifold} \implies cld \text{Diff}^r(M)_0 \leq 10

T. Tsuboi \quad M: \text{a closed } n\text{-manifold} \quad (2008, 2012)

(1) \quad n = 2m + 1: \quad cld \text{Diff}^r(M)_0 \leq 4

(2) \quad n = 2m:

(i) \quad \exists \text{Handle decomp. without } m\text{-handles} \implies cld \text{Diff}^r(M)_0 \leq 3

(ii) \quad m \geq 3, \quad \exists \text{Triangulation with } \# m\text{-simplices} \leq k

\implies cld \text{Diff}^r(M)_0 \leq 4k + 11

T. Tsuboi \quad M: \text{a closed connected } n\text{-manifold} \quad (2009, 2012)

\text{Diff}^r(M)_0 : \text{uniformly simple \quad if}

(1) \quad n \neq 2, 4 \quad \text{or} \quad (2) \quad n = 2, 4 \quad \exists \text{Handle decomp. without } m\text{-handles} \quad (n = 2m)

T. Rybicki \quad (2011)

M = \text{Int } W \quad (W: \text{a compact } n\text{-manifold with } \partial \neq \emptyset)

M: \text{portable} \implies \text{Diff}^r(M)_0 : \text{bounded}
§2. Conjugation-invariant norms (BIP).

$G$: a group

(1) an extended conjugation-invariant norm on $G$

\[ q : G \rightarrow [0, \infty] \text{ s.t. } \begin{align*}
(i) & \quad q(g) = 0 \text{ iff } g = e \\
(ii) & \quad q(g^{-1}) = q(g) \\
(iii) & \quad q(gh) \leq q(g) + q(h) \\
(iv) & \quad q(hgh^{-1}) = q(g)
\end{align*} \]

\( \circ \ G \supset A \leadsto qdA := \sup \{q(g) \mid g \in A\} \).

(2) a conjugation-invariant norm on $G$

= an extended conj.-invariant norm on $G$ with values in $[0, \infty)$

\( \circ \ G : \text{bounded} \iff \text{any conj.-invariant norm on } G \text{ is bounded} \)

(3) $S \subset G$: symmetric, conjugation-invariant

\( G \triangleright N(S) = S^\infty \equiv \bigcup_{k=0}^\infty S^k \)

\((S = S^{-1}) \quad (gSg^{-1} = S \ (\forall g \in G))\)

\( q(G,S) : G \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\} \text{ is defined by} \)

\[
q(G,S)(g) := \begin{cases} 
\min\{k \in \mathbb{Z}_{\geq 0} \mid g = g_1 \cdots g_k \text{ for } g_1, \cdots, g_k \in S\} & (g \in N(S)) \\
\infty & (g \in G - N(S)) \). 
\end{cases}
\]
[I] Conjugation-generated norm $\nu_g : G \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$

(1) $g \in G$ $\quad C(g) :=$ the conjugacy class of $g$ in $G$

$C_g := C(g) \cup C(g^{-1}) \subset G$ : symmetric, conjugation invariant

(2) $\nu_g := q_{(G,C_g)}$ $\quad N(g) = N(C_g) = C_g^\infty$

$\circ \nu_g(f) \leq k \iff f = \text{a product of at most } k \text{ conjugates of } g \text{ or } g^{-1}$

(3) $G : \text{uniformly simple} \iff \exists k \in \mathbb{Z}_{\geq 0} \text{ s.t. } \nu_g \leq k \quad (\forall g \in G - \{e\})$

(4) $\nu_g$ is bounded for some $g \in G - \{e\} \implies G : \text{bounded}$

[II] Commutator length $cl_G : G \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$

(1) $G^c = \{[a,b] \mid a, b \in G\}$ : symmetric, conj.-invariant in $G$

$\circ [G, G] = N(G^c) = (G^c)^\infty$

$cl_G := q_{(G,G^c)}$ $\quad cld_G, cld_G A \equiv cld(A,G) \quad (A \subset G)$

$\circ cl(g) \leq k$ in $G \iff cl_G(g) \leq k$

(2) $G : \text{uniformly perfect} \iff \text{cld}_G < \infty$

i.e., $\exists k \in \mathbb{Z}_{\geq 0} \text{ s.t. } G \ni \forall g = g_1 \cdots g_k \quad (\exists g_1, \cdots, g_k \in G^c)$
Commutator length supported in $n$-balls in diffeomorphism groups

$clb^f, clb^d : \text{Diff}^r(M, \partial M \cup C)_0 \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \quad (C \subset M : \text{a closed subset})$

(1) $\mathcal{B} = \mathcal{B}^r_f(M, C), \mathcal{B}^r_d(M, C)$

$\mathcal{B}^r_f(M, C) := \text{the collection of all finite disjoint unions } D \text{ of } C^r \text{ } n\text{-balls in } M \text{ s.t. } D \subset \text{Int } M - C$

$\mathcal{B}^r_d(M, C) := \text{the collection of all discrete unions } D \text{ of } C^r \text{ } n\text{-balls in } M \text{ s.t. } D \subset \text{Int } M - C$

(2) $G \equiv \text{Diff}^r(M, \partial M \cup C)_0$

$S := \bigcup \{\text{Diff}^r(M, M_D)_0^c \mid D \in \mathcal{B}\} \quad (M_A := M - \text{Int}_MA \quad (A \subset M))$

$S : \text{symmetric, conjugation-invariant in } G$

$clb^f, clb^d := q_{(G, S)} \quad clb^f d, \quad clb^d d$

$\circ \quad M - C : \text{relatively compact in } M$

$\implies \quad \mathcal{B}^r_f(M, C) = \mathcal{B}^r_d(M, C), \quad clb^f = clb^d \quad \text{on } G$
Relations between $\nu_g$ and $clb^f$, $clb^d$

$C(X) :=$ the set of connected components of a top. space $X$

**Definition.** $M :$ an $n$-manifold possibly with $\partial$, $g \in \text{Diff}^r(M)$

1. $g :$ component-wise non-trivial $\iff g|_U \neq \text{id}_U \quad (\forall U \in C(M))$
2. $g :$ component-wise end-non-trivial $\iff$
   - (i) $g :$ component-wise non-trivial
   - (ii) $g|_V \neq \text{id}_V \quad \forall (U, K, V)$ with $U \in C(M) - K(M), K \in K(U)$
     $V \in C(U - K) :$ not rel. compact

**Fact.** $M :$ an $n$-manifold possibly with $\partial$

1. $\nu_g \leq 4clb^f$ in $\text{Diff}^r_c(M, \partial)_0$
   
   if $g \in \text{Diff}^r_c(M, \partial)_0 :$ component-wise non-trivial $(C(M) :$ a finite set)$

2. $\nu_g \leq 4clb^d$ in $\text{Diff}^r(M, \partial)_0$
   
   if $g \in \text{Diff}^r(M, \partial)_0 :$ component-wise end-non-trivial

* Upper bound for $cl$, $clb^f$, $clb^d \implies$ Uniform Perfectness
  Boundedness
  Uniform Simplicity
§3. Main Results.

1 \leq r \leq \infty, \ r \neq n + 1

**Theorem I.** \( n = 2m + 1 \).

1. \( M \) : a compact \( n \)-manifold possibly with \( \partial \) \( \implies \ \cld \text{Diff}^r(M, \partial)_0 \leq 4 \)

2. \( M \) : an open \( n \)-manifold \( \implies \ \cld \text{Diff}^r(M)_0 \leq 8, \ \cld \text{Diff}^r_c(M)_0 \leq 4 \)

**Theorem II.** \( n = 2m \).

1. \( M \) : a compact \( n \)-manifold possibly with \( \partial, \ m \geq 3 \)

   \[ \cld \text{Diff}^r(M, \partial)_0 \leq 2k + 7 \]

   if \( \exists \) Triangulation s.t. \( \# \{ m \text{-simplices not in} \ \partial M \} \leq k \)

2. \( M \) : an \( n \)-manifold without boundary

   (i) \( \cld \text{Diff}^r(M)_0 \leq 6 \) and \( \cld \text{Diff}^r_c(M)_0 \leq 3 \)

   if \( \exists \) Handle decomp. without \( m \)-handles

   (ii) \( \cld \text{Diff}^r(M)_0 \leq 2k + 10 \) and \( \cld \text{Diff}^r_c(M)_0 \leq 2k + 7 \)

   if \( m \geq 3, \ \exists \) Handle decomp. \( \mathcal{H} \) s.t. \( \# m \)-handles \( \leq k \)

   each closed \( m \)-cell of \( P_\mathcal{H} \) has \( \text{SDP} \) for \( P^{(m)}_\mathcal{H} \).

\( \circ \ \text{SDP} = \text{Strong Displacement Property} \)
Theorem III. \( \pi: \tilde{M} \to M \): a \( C^\infty \) covering space

- \( M \): a closed \( 2m \)-manifold \( (m \geq 3) \)
- \( \exists \) Triangulation of \( M \) s.t. \# \( m \)-simplices \( \leq k \)
- or \( \exists \) Handle decomp. \( \mathcal{H} \) of \( M \) s.t. \# \( m \)-handles \( \leq k \)

Each closed \( m \)-cell of \( P_{\mathcal{H}} \) has SDP for \( P_{\mathcal{H}}^{(m)} \)

\[ \implies cld \text{ Diff}^r(\tilde{M})_0 \leq 4k + 14 \quad \text{and} \quad cld \text{ Diff}^r_c(\tilde{M})_0 \leq 2k + 7. \]

Theorem IV.

\( M = \bigoplus_{i=1}^{\infty} N \): an infinite connected sum of a closed \( 2m \)-manifold \( N \)

(or an infinite sum of finitely many compact \( 2m \)-manifolds)

\[ \implies cld \text{ Diff}^r(M)_0 < \infty \quad \text{and} \quad cld \text{ Diff}^r_c(M)_0 < \infty. \]

Theorem V. \( M = \text{Int } W \) \( (W \): a compact \( n \)-manifold with \( \partial \neq \emptyset \) )

\[ \implies cld \text{ Diff}^r(M)_0 \leq \max\{cld \text{ Diff}^r(W, \partial)_0, 2\} + 2. \]

\( \circ \) \( \text{Diff}^r(M)_0 \): uniformly perfect for \( n \neq 2, 4 \)
Theorem VI.

[1] $M$: a compact connected $n$-manifold possibly with $\partial$, $n \neq 2, 4$

$\implies$ $\text{Diff}^r(M, \partial)_0$: uniformly simple


$\text{Diff}^r(M)_0$: bounded and $\text{Diff}^r_c(M)_0$: uniformly simple

in the following cases:

(1) $n = 2m + 1$

(2) $n = 2m$, $M$ satisfies one of the following conditions:

(i) $\exists$ Handle decomp. without $m$-handles

for $m \geq 3$

(ii) $\exists$ Handle decomp. $\mathcal{H}$ s.t. $\# m$-handles $< \infty$

each closed $m$-cell of $P_{\mathcal{H}}$ has SDP for itself in $M$

(iii) $M$: a covering space over a closed $2m$-manifold

(iv) $M$: an infinite sum of finitely many compact $2m$-manifolds

[3] $M = \text{Int} W$ ($W$: a compact $n$-manifold with $\partial \neq \emptyset$)

$\text{Diff}^r(W, \partial)_0$: bounded $\implies$ $\text{Diff}^r(M)_0$: bounded
§4. Absorption / displacement property

$M$: an $n$-manifold possibly with $\partial$  
$O(M), \mathcal{F}(M), \mathcal{K}(M)$  
$O \in \mathcal{O}(M), K \in \mathcal{K}(M), L \in \mathcal{F}(M)$  
$\mathcal{P}$: a condition for $\varphi \in \text{Diff}_c(M, M_O)_0$.

**Definition I. (Absorption property)**

(1) $C \in \mathcal{K}(O)$:
   (i) $C$ is absorbed to $K$ in $O$ with $\mathcal{P}$
       $$\iff \exists \varphi \in \text{Diff}_c(M, M_O)_0 \text{ s.t. } \varphi(C) \subset K, \varphi : \mathcal{P}$$
   (ii) $C$ is weakly absorbed to $K$ in $O$ with $\mathcal{P}$
       $$\iff C \text{ is absorbed to "any neighborhood of } K" \text{ in } O \text{ with } \mathcal{P}.$$  

(2) $K$ has the (weak) absorption property in $O$ with $\mathcal{P}$

$$\iff \forall C \in \mathcal{K}(O) \text{ is (weakly) absorbed to } K \text{ in } O \text{ with } \mathcal{P}.$$  

(3) When $K \in \mathcal{K}(O)$:
   $K$ has the (weak) neighborhood absorption property in $O$ with $\mathcal{P}$

$$\iff \exists \text{ a compact nbd of } K \text{ in } O \text{ is (weakly) absorbed to } K \text{ in } O \text{ with } \mathcal{P}.$$
Definition II. (Displacement property)

(1) $K$ is displaceable from $L$ in $O$

$$
\iff \exists \psi \in \text{Diff}_c(M, M_O)_0 \text{ s.t. } \psi(K) \cap L = \emptyset
$$

(2) $K$ is strongly displaceable from $L$

$$
\iff K \text{ is displaceable from } L \text{ in any open neighborhood of } K \text{ in } M.
$$

Example. $K$ is strongly displaceable from $L$ in $M$ in the following cases:

(1) $K$ has arbitrarily small open $n$-disk nbds $U$ in $M$ with $U \not\subset L$.

(2) $K \subset \text{Int } M$ : a compact $k$-dim stratified subset

$L \subset M$ : an $\ell$-dim stratified subset

(i) $k + \ell < n$

(ii) $k + \ell = n$ : $\text{Cl}_M(K - K^{(k-1)})$ : str. disp. from $\text{Cl}_M(L - L^{(\ell-1)})$ in $M$

(3) $L \subset M$ : a submanifold, $K \in \mathcal{K}(L)$

The normal bundle of $L$ in $M$ admits a non-vanishing section over $K$. 
Theorem II'. \( n = 2m \).

(2) \( M \) : an \( n \)-manifold without boundary

\[ cld \text{Diff}^r(M)_0 \leq 3k + 8 \quad \text{and} \quad cld \text{Diff}^r_c(M)_0 \leq 3k + 5 \]

if \( m \geq 3 \), \( \exists \) Handle decomp. \( \mathcal{H} \) s.t. \# \( m \)-handles \( \leq k \)

each closed \( m \)-cell of \( P_\mathcal{H} \) has **SDP for itself** in \( M \).

Example in Closed manifold case.

\( M = S^m \times S^m \) : the product of two \( m \)-spheres

\( S^m \) has a natural handle decompostion with one 0-handle and one \( m \)-handle

\( \mathcal{H} \) : the product handle decompostion of \( M \)

(one 0-handle, two \( m \)-handles and one \( 2m \)-handle)

\( P_\mathcal{H} \) : the core complex of \( \mathcal{H} \)

two open \( m \)-cells \( \sigma_j \) (\( j = 1, 2 \))

\( Cl_M \sigma_j \subset M \) (\( j = 1, 2 \)) : smooth \( m \)-spheres with a trivial normal bundle

each \( Cl_M \sigma_j \) (\( j = 1, 2 \)) is strongly displaceable from itself in \( M \)

\( \therefore cld \text{Diff}^r(M)_0 \leq 11 \quad \text{if} \quad m \geq 3, \quad 1 \leq r \leq \infty, \quad r \neq 2m + 1. \)

\( \circ Cl_M \sigma_j \) is not displaceable from \( Cl_M(\sigma_1 \cup \sigma_2) \).
§5. Estimation on $cl, clb^f, clb^d$

[1] Basic Strategy due to BIP & T. Tsuboi

The case for closed $n$-manifolds + handle decompositions $\mathcal{H}$

( s.t. $\mathcal{P}_\mathcal{H}$ : Cell complex in $2m$-dim )

[2] Extension of Basic Strategy :

In the compact case :

The case for compact manifolds with $\partial$ + triangulations

The case for compact $n$-submanifolds

in open $n$-manifolds + handle decompositions

In the open manifold case :

Improvements + New Ideas :

Arguments in Basic Strategy — effective even in the closed manifold case

(1) Separate the absorption/displacement property from the related arguments.

(2) Improvements of factorizations of isotopies
(3) $2m$-dim case:

Grouping of closed $m$-cells in handle decompositions

$m$-simplices in triangulations

(under some displacement conditions)

(2), (3) $\leadsto$ Finer estimates of $cl$ and $clb^f$

For compact manifolds with $\partial$:

(1) Extension of basic factorization lemmas to the $\partial$-case

(under the absorption/displacement condition)

(2) Use triangulations, the (double mapping) cylinder structure between complementary full subcomplexes and the flows induced by this cylinder structure.

$\circ$ Can not use the usual handle decompositions.

For compact $n$-submanifolds

in an open $n$-manifold $M$ with a handle decomposition $\mathcal{H}$
(1) Use triples $N \subseteq N_1 \subseteq N_2$ of compact $n$-submanifolds s.t.

$$N_1 : \mathcal{H}\text{-saturated}, \quad N_2 : \mathcal{H}^\ast\text{-saturated}.$$ 

(2) Obtain the relative estimates

$$cld \left( \text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_{N_2})_0 \right)$$

○ Can not apply Basic Strategy to $N_1$ directly.

**For open manifolds**

(1) Factorization of isotopies on open manifolds — Reduction to Compact case

(2) $2m$-dim case:

Grouping of closed $m$-cells in handle decompositions

$m$-simplices in triangulations

(under some displacement conditions)

Take a finite cover of infinitely many closed $m$-cells / $m$-simplices

in the following cases:

(i) a covering space of a closed $2m$-manifold

(ii) an infinite sum of finitely many compact $2m$-manifolds.

(3) Introduce $clb^d$ to deduce the boundedness of $\text{Diff}^r(M)_0$. 
Factorization of isotopies on open manifolds:

\( M \) : an open \( n \)-manifold

\( \forall F \in \text{Isot}^r(M)_0 \)

\( \exists \) a factorization \( F = GH \) s.t.

\[
\operatorname{supp} G \subset \bigcup_{k=1}^{\infty} L_k : \text{a discrete union of compact } n\text{-submfd of } M
\]

\[
\operatorname{supp} H \subset \bigcup_{k=1}^{\infty} N_k : \text{a discrete union of compact } n\text{-submfd of } M
\]
References.

D. Burago, S. Ivanov and L. Polterovich (BIP)

T. Tsuboi

K. Fukui, T. Rybicki, T. Yagasaki (FRY)
End of Talk

Thank you very much for your attention!