Topological Properties of Diffeomorphism Groups of Non-Compact Manifolds

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The diffeomorphism group \( \mathcal{D}(M) \) of a non-compact \( C^\infty \) \( n \)-manifold \( M \) admits two typical topologies: the compact-open \( C^\infty \) topology and the Whitney \( C^\infty \) topology. In this talk we discuss local and global properties of these topologies. We will see that the local topological types are represented by the following correspondence:

- Compact-Open \( C^\infty \) Topology \( \leftrightarrow \) Tychonoff Products \( \cdot \) Weak Products of \( \ell_2 \)
- Whitney \( C^\infty \) Topology \( \leftrightarrow \) Box Products \( \cdot \) Small Box Products of \( \ell_2 \)

§.1. Notations.

\( M \) : Connected \( C^\infty \) \( n \)-manifold.

\( \mathcal{D}(M) = \) the group of diffeomorphisms of \( M \)

\( \mathcal{D}(M; \omega) = \) the subgroup of \( \omega \)-preserving diffeomorphisms of \( M \) (when \( M \) has a volume form \( \omega \))

For any subgroup \( \mathcal{G}(M) \) of \( \mathcal{D}(M) \),

(i) the scripts \( \mathcal{G}^+(M) \) and \( \mathcal{G}^c(M) \) denote “orientation - preserving” and “compact support”.

(ii) When \( \mathcal{G}(M) \) is endowed with a topology,

\( \mathcal{G}(M)_0 \), \( \mathcal{G}(M)_1 \) = the connected component and path component of \( \text{id}_M \) in \( \mathcal{G}(M) \)

\( \mathcal{G}^c(M)_1^* = \{ h \in \mathcal{G}^c(M)_1 \mid \exists \text{ a path } h_t : h \simeq \text{id}_M \text{ in } \mathcal{G}^c(M) \text{ with common compact support} \} \)

§.2. Properties of Compact-Open \( C^\infty \) Topology.

In this section the group \( \mathcal{D}(M) \) and its subgroups are endowed with **Compact-Open \( C^\infty \) Topology**.

When \( M \) is compact, \( \mathcal{D}(M) \) is a Fréchet manifold (hence a topological \( l_2 \)-manifold) and “the parametrized version of Moser’s theorem for volume forms” induces the following diagram:

\[
\mathcal{D}^+(M) \supset \mathcal{D}(M)_0 \\
\cup \text{ SDR} \cup \cup \text{ SDR} \\
\mathcal{D}(M; \omega) \supset \mathcal{D}(M; \omega)_0
\]

When \( M \) is Non-Compact, from the parametrized version of Moser’s theorem for non-compact manifolds and the existence of a continuous section of the end-charge homomorphism \( c_0^\omega : \mathcal{D}(M; \omega)_0 \longrightarrow \mathcal{S}(M; \omega) \) we have the following diagram:

**Theorem 2.1.**

\[
\mathcal{D}^+(M) \supset \mathcal{D}(M)_0 \supset \mathcal{D}^c(M)_0 \supset \mathcal{D}^c(M)_1^* \\
\cup \text{ SDR} \cup \cup \cup \\
\mathcal{D}(M; \omega) \supset \mathcal{D}(M; \omega)_0 \supset \text{ker } c_0^\omega \supset \mathcal{D}^c(M; \omega)_0 \supset \mathcal{D}^c(M; \omega)_1^* \\
\text{ SDR}
\]

**Remaining Problems.**

[1] Homotopy / Topological Type of \( \mathcal{D}(M)_0 \) and \( \mathcal{D}(M; \omega)_0 \)

[2] Relations : \( \mathcal{D}(M)_0 \supset \mathcal{D}^c(M)_1^* \), \( \text{ker } c_0^\omega \supset \mathcal{D}^c(M; \omega)_1^* \)

In \( n = 2 \) we can answer these questions.
The 2-dim case:

\[ M : \text{a non-compact connected } C^\infty \text{ 2-manifold without boundary.} \]

Exceptional Case — Plane, Open Möbius Band and Open Anulus

Generic Case — all other cases

[1] Homotopy / Topological Type of \( D(M)_0 \) and \( D(M; \omega)_0 \)

**Theorem 2.2.**

(1) \( D(M)_0, D(M; \omega)_0 \) : Topological \( \ell_2 \)-manifold

(2) (i) Generic Case : \( D(M)_0 \simeq D(M; \omega)_0 \simeq * \)

(2) (ii) Exceptional Case : \( D(M)_0 \simeq D(M; \omega)_0 \simeq S^1 \)

**[2] Subgroups** \( D^c(M)_1^* \) and \( D^c(M; \omega)_1^* \)

**Remark on** \( D^c(M)_1^* \)

(1) Each \( h \in D^c(M)_1^* \) is isotopic to \( \text{id}_M \) with common compact support.

(2) A path in \( D^c(M)_1^* \) need not have common compact support.

\[ \circ \ h_t \rightarrow \text{id}_M \text{ in } D^c(M)_1^* \text{ if supp } h_t \rightarrow \infty. \]

**Example 1.** \( M = \mathbb{R}^2 \)

(1) \( D(\mathbb{R}^2)_0 \simeq S^1 \) : A homotopy equivalence is induced from **Loop of \( \theta \) rotations** : \( \varphi(\theta) \) (\( \theta \in [0, 2\pi] \))

(2) \( D^c(\mathbb{R}^2)_1^* = D^c(\mathbb{R}^2)_0 : \text{a homotopy equivalence} \)

(i) The following claim is **false**: \( D^c(\mathbb{R}^2)_0 \simeq * \) by \( D^c(S^2) \simeq * \)

(ii) \( \exists \text{ Deformation of loops } : \varphi_t(\theta) \) (\( t \in [0, 1] \)) of \( \varphi_0(\theta) \equiv \varphi(\theta) \) s.t. \( \varphi_\infty(\theta) \in D^c(\mathbb{R}^2) \) (\( 0 < t \leq 1 \)).

\( \circ \) We can take \( \varphi_t(\theta) \) (\( \theta \in [0, 2\pi] \)) as **Loop of Truncated \( \theta \) rotations**.

\( \circ \) **Level of Truncation** \( r_t(\theta) \) (\( 0 < t \leq 1 \)) need to satisfy the following conditions:

(i) \( r_t(\theta) \rightarrow \infty \) as \( \theta \rightarrow 2\pi \) (for each \( t > 0 \))

(ii) \( r_t(\theta) \rightarrow \infty \) uniformly in \( \theta \) as \( t \rightarrow 0 \)

**Example 2.** \( M = \text{Open Anulus} \quad h : \text{Dehn Twist on } M \) along the center circle of \( M \)

(1) \( h \in D(M)_1 \setminus D_c(M)_1^* \)

(2) \( \exists \) a path \( h_t : h \simeq \text{id}_M \) in \( D(\omega)_0 \) s.t. \( h_t \in D_c(M)_1^* \) (\( 0 < t \leq 1 \))

(Introduce **Reverse Dehn Twist** from \( \infty \))

**Definition 2.1.** \( A \subset X : \text{Homotopy Dense (HD)} \)

\[ \iff \exists \varphi_t : X \rightarrow X : \text{Homotopy s.t. } \varphi_0 = \text{id}_X, \varphi_t(X) \subset A \text{ (0 < t \leq 1)} \]

**Theorem 2.3.** \( D(M)_0 \supset D^c(M)_1^* : \text{Homotopy Dense} \)

\( \circ \ D(M)_0 \supset D^c(M)_0 = D^c(M)_1 \supset D^c(M)_1^* : \text{Homotopy Equi.} \)

\[ (\prod {}^{\omega} \ell_2, \sum {}^{\omega} \ell_2) : \sum {}^{\omega} \ell_2 = \{ (x_i) \in \prod {}^{\omega} \ell_2 | x_i = 0 \text{ except for finitely many } i \} \quad (\prod {}^{\omega} \ell_2 \cong \ell_2) \]

**Corollary 2.1.** \( \mathcal{E} = D^c(M)_0 \) or \( D^c(M)_1^* \)

(1) \( D(M)_0, \mathcal{E} ) : \text{Topological } (\prod {}^{\omega} \ell_2, \sum {}^{\omega} \ell_2)\text{-manifold} \)
Theorem 3.3. \[ \ker c_0^\omega \supset D^c(M; \omega)_1^* : \text{Homotopy Dense} \]
\[ \circ \ker c_0^\omega \supset D^c(M; \omega)_0 = D^c(M; \omega)_1 \supset D^c(M; \omega)_1^* : \text{Homotopy Equi.} \]

Theorem 2.4. \[ \ker c_0^\omega \supset D^c(M; \omega)_1^* : \text{Homotopy Dense} \]
\[ \circ \ker c_0^\omega \supset D^c(M; \omega)_0 = D^c(M; \omega)_1 \supset D^c(M; \omega)_1^* : \text{Homotopy Equi.} \]

§3. Properties of Whitney $C^\infty$-Topology. (Joint Work with T. Banakh, K. Mine and K. Sakai)

$M$ : a connected $C^\infty$ $n$-manifold without boundary

$D^c(M)^w = D^c(M)$ with Whitney $C^\infty$-topology

- $h \in D^c(M)^w$ has basic neighborhoods of the following form:

$$\bigcap_{\lambda \in A} U(h, (U_\lambda, x_\lambda), (V_\lambda, y_\lambda), K_\lambda, r_\lambda, \varepsilon_\lambda)$$

where $\{U_\lambda\}_{\lambda \in A}$ is locally finite in $M$

When $M$ is Compact, Whitney $C^\infty$-Top $=$ Compact-Open $C^\infty$-Top

When $M$ is Non-Compact, Whitney $C^\infty$-Top is Too Strong (Compact-Open $C^\infty$-Top is Too Weak):

1. $D^c(M)_0^w = D^c(M)_1^* \subset D^c(M)$ (as Sets)
2. Any compact subset (for example, any path) in $D^c(M)^w$ has Common Compact Support.

[1] Local Top Type of $D^c(M)^w$ and $D^c(M)^w$

Models: $(\Box^\omega, \Box^\omega)l_2 : \text{Pair of Box Product and Small Box Product of } l_2$

$$\mathbb{R}^\infty := \lim_{\to n} \mathbb{R}^n \approx \Box^\omega l_2$$

Theorem 3.1.

1. $D^c(M)^w : \text{Paracompact } (\ell_2 \times \mathbb{R}^\infty)$-manifold
2. $D^c(M)^w_0 \subset D^c(M)^w_0 : \text{Open Normal Subgroup}$
3. $\mathcal{M}_c(M) := D^c(M)^w_0 / D^c(M)^w_0$ (Discrete Countable Group)
4. $D^c(M)^w \approx D^c(M)^w_0 \times \mathcal{M}_c(M)$ (as Top Spaces)
5. $(M_i)_{i \in \mathbb{N}} : \text{Sequence of Compact } C^\infty n\text{-submanifolds of } M \text{ s.t. } M_i \subset \text{Int}_M M_{i+1}, \ M = \cup_i M_i \ G(M_i) := \{h \in D^c(M)^w \mid \text{supp } h \subset M_i\} \implies D^c(M)^w = \underleftarrow{\lim_{i \to}} G(M_i)$ (Direct Limit in Category of Top Groups)

- $D^c(M)^w$ with Direct Limit Top : Not Top Group

Theorem 3.2. \[ (D(M)^w, D^c(M)^w) \approx (\Box^\omega, \Box^\omega)l_2 \text{ at } \operatorname{id}_M \text{ locally} \]

[2] Global Top Type of $D^c(M)^w$

Theorem 3.3.

1. $n = 1 : (D(\mathbb{R})^w, D^c(\mathbb{R})^w) \approx (\Box^\omega, \Box^\omega)l_2$
2. $n = 2 : D(M)^w_0 \approx \ell_2 \times \mathbb{R}^\infty$
3. $n = 3 : M : \text{Orientable, Irreducible} \implies D(M)^w_0 \approx \ell_2 \times \mathbb{R}^\infty$
4. $X : \text{Compact Connected } C^\infty n\text{-manifold with Boundary}$
   \[ M = \text{Int } X \implies D(M)^w_0 \approx D(X, \partial X)^w_0 \times \mathbb{R}^\infty \]
\[ \mathcal{M}_c(M) = \mathcal{D}^c(M)^w / \mathcal{D}(M)_0^w \quad \text{in} \ n = 2 \]

\[ S : \text{Connected 2-manifold possibly with Boundary} \]

(1) \( S : \text{Exceptional} \quad \text{def} \iff S \approx N - K, \text{ where } N = \text{Annulus, Disk or Möbius band} \)
\[ K = \text{Non-empty Compact subset of One boundary circle of } X \]

(2) \( S : \text{Semi-Finite Type} \quad \text{def} \iff S \approx N - (F \cup K) \text{ s.t. } N : \text{Compact Connected 2-manifold} \)
\[ F \subset N \setminus \partial N : \text{Finite subset} \]
\[ K \subset \partial N : \text{Compact subset} \]

\[ \text{equiv} \iff \pi_1(S) : \text{finitely presented} \iff H_1(S;\mathbb{Z}) : \text{finitely generated} \]

**Proposition 3.1.** The Following Conditions are Equivalent:

(1) \( \mathcal{M}_c(S) = \{1\} \)

(2) \( \mathcal{M}_c(S) : \text{Torsion Group} \)

(3) \( \text{asdim} \mathcal{M}_c(S) = 0 \)

(4) \( S : \text{Exceptional} \)

**Proposition 3.2.** The Following Conditions are Equivalent:

(1) \( \mathcal{M}_c(S) : \text{finitely generated (or finitely presented)} \)

(2) \( r_z \mathcal{M}_c(S) < \infty \)

(3) \( \text{asdim} \mathcal{M}_c(S) < \infty \)

(4) \( S : \text{Semi-Finite Type} \quad (S : \text{Not Semi-Finite Type} \implies \oplus^\infty \mathbb{Z} \subset \mathcal{M}_c(S)) \)

**References.** This talk is based upon the following papers and preprints:


   Homeomorphism and diffeomorphism groups of non-compact manifolds with the Whitney topology, Topology Proceedings, 37 (2011) 61 - 93.

[3] T. Banakh and T. Yagasaki,

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