Diffeomorphism groups of non-compact manifolds
dewed with the Whitney $C^\infty$-topology

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2015 Math. Society of Japan
Annual Meeting, Topology Section
March. 22, 2015, Meiji Univ.
§1. Diffeomorphism Groups with Whitney $C^\infty$-topology

$M$ : Non-compact $C^\infty$ n-manifold (without boundary)

$\mathcal{D}(M)$ : Diffeo Group of $M$ (Whitney $C^\infty$-topology) — Top group

$\mathcal{D}_0(M)$ : the identity connected component of $\mathcal{D}(M)$

$\mathcal{D}_c(M) \subset \mathcal{D}(M)$ : Subgroup of Diffeo’s with compact support

Basic Property. (BMSY, 2011)

(1) $\mathcal{D}_c(M)$ : Paracompact $\mathcal{D}_0(M) \subset \mathcal{D}_c(M)$ : Open normal subgroup

(2) $(M_i)_{i \in \mathbb{N}}$ : Compact n-submfds of $M$ s.t. $M_i \subset \text{Int}_MM_{i+1}$, $M = \bigcup_i M_i$

$G(M_i) := \{h \in \mathcal{D}_c(M) \mid \text{supp } h \subset M_i\} < \mathcal{D}_c(M)$ : Closed subgroup

$G(M_1) \subset G(M_2) \subset \cdots \quad \mathcal{D}_c(M) = \bigcup_i G(M_i)$

(i) $\mathcal{D}_c(M) = \varprojlim_i G(M_i)$ (Direct limit in Category of Top Groups)

(ii) $G(M_i)$ : Top manifold modeled on $\text{Infinite-dim separable Fréchet space } \approx l_2$

Problem. Global Top Type of $\mathcal{D}_c(M)$ ?
**LF spaces** = Direct Limits of Fréchet spaces in Category of Top vector spaces

Top classification of LF spaces  (P. Mankiewicz 1974)

Infinite-dim separable LF spaces (up to homeo.)

1. \( \mathbb{R}^\omega \equiv \text{dir lim} \{ \mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots \} \approx \Box^\omega \mathbb{R} \)

2. \( l_2 \)

3. \( l_2 \times \mathbb{R}^\omega \approx \Box^\omega l_2 \)

**Box product · Small box product**

**Definition.** \( X \quad \omega = \{0, 1, 2, \cdots \} \)

1. Box product of \( X \) : \( \Box^\omega X = \prod_{n \in \omega} X \)

   Box Topology : \( \prod_n U_n \quad (U_n \subset X : \text{Open subset}) \)

2. Small box product of \((X, \ast)\) : \( \Box^\omega X \subset \Box^\omega X \)

   Subspace of finite sequences : \( (x_0, x_1, \ldots, x_i, \ast, \ast, \ldots) \)

   Box product \( \prod_i X_i, \quad \text{Small box product} \quad \Box^\omega_i (X_i, \ast_i) \)
Top. Type

1-dim case \((\mathcal{D}(\mathbb{R}), \mathcal{D}_c(\mathbb{R})) \approx (\Box^\omega l_2, \Box^\omega l_2)\) (BY, 2010)

\(n\)-dim case \((\mathcal{D}(M), \mathcal{D}_c(M)) \approx_\ell (\Box^\omega l_2, \Box^\omega l_2)\) (BMSY, 2011)

\(*) \mathcal{D}_c(M) \approx_\ell \Box^\omega l_2 \approx l_2 \times \mathbb{R}^\infty\)

Thm. \(\mathcal{D}_c(M) \approx\) Open subset of \(l_2 \times \mathbb{R}^\infty\) (BY 2015)

Cor. \(\mathcal{D}_c(M) \approx L \times \mathbb{R}^\infty\) for some \(l_2\)-manifold \(L\) \((L \approx \mathcal{D}_c(M))\)

(1) \(M\) : Connected

\(\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty\) if \(n = 1, 2\) or

\(n = 3\) and \(M\) : orientable, irreducible

(2) \(\mathcal{D}_0(M) \approx \mathcal{D}_0(N; \partial N) \times \mathbb{R}^\infty\)

if \(N\) : Compact Connected \(C^\infty\) \(n\)-manifold with Boundary

\(M = \text{Int} N\)
§4. Results on Top Groups and Towers of Subgroups

\( G \): Top group \((e \text{: the identity element of } G)\)

\( G_n \) \((n \in \omega)\): Tower of Closed subgroups of \( G \)

\((G_0 \subset G_1 \subset G_2 \subset \cdots, G = \bigcup_n G_n)\)

\( p : \square_n(G_n,e) \rightarrow G : p(x_0, x_1, \ldots, x_k, e, e, \ldots) = x_k \cdots x_1 x_0 \)

* \( p \): continuous, surjective


\( p : \text{open} \implies G = \varprojlim G_n \) (in Category of Top Groups)


(i) \( p : \text{open} \) \hspace{1cm} (ii) \( G_n \rightarrow G_n/G_{n-1} \) admits a global section \( s_n \).

\[ \implies s = \square_n s_n \stackrel{\square_n G_n}{\rightarrow} \square_n(G_n/G_{n-1}) \stackrel{ps}{\sim} G \]

Study of Top LF-manifolds and Direct limit of Uniform spaces

**Criterion for open subsets of** $\ell_2 \times \mathbb{R}^\infty$ (BMRSY, 2013)

(i) $G$: Non-metrizable

(ii) $G_n$: separable $\ell_2$-manifold

(iii) $p: \Box_n G_n \to G$: open

(iv) $G_n \to G_n/G_{n-1}$ has a local section

(v) $G_n/G_{n-1}$ is an $\ell_2$-manifold

(more generally, each $Z$-point of $G_n/G_{n-1}$ is a strong $Z$-point).

$\implies G \approx$ Open subset of $\ell_2 \times \mathbb{R}^\infty$
References.

BMRSY = T. Banakh, K. Mine, D. Repovš, K. Sakai, T. Yagasaki


Thank you very much for your attention!