Groups of uniform homeomorphisms of covering spaces

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§.1. Main Problem

Local and Global Homotopical / Topological Properties of
Groups of Uniform Homeomorphisms of Non-Compact Metric Manifolds

\((X, d), (Y, \varrho)\) : metric spaces

**Definition 1.1.** \(f : (X, d) \rightarrow (Y, \varrho)\)

1. \(f : \) uniformly continuous \(\iff\)
   \[
   \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } x, x' \in X, \ d(x, x') < \delta \implies \varrho(f(x), f(x')) < \varepsilon
   \]
2. \(f : \) a uniform homeomorphism
   \[
   \iff f : \text{bijective} \quad f, f^{-1} : \text{uniformly continuous}
   \]
3. a uniform embedding = a uniform homeomorphism onto its image

\(f, g : (X, d) \rightarrow (Y, \varrho)\)

the sup-metric : \(\varrho(f, g) = \sup \{ \varrho(f(x), g(x)) \mid x \in X \} \in [0, \infty] \)

**the uniform topology** = the topology induced from the sup-metric
$\mathcal{H}^u(X, d) = \text{Group of uniform homeomorphisms of } (X, d) \text{ onto itself}$

$\bigcup \text{ open (} \& \text{ closed) (the uniform topology)}$

$\mathcal{H}^u(X, d)_b = \{ h \in \mathcal{H}^u(X, d) \mid d(h, \text{id}_X) < \infty \}$

$\bigcup$

$\mathcal{H}^u(X, d)_0 = \text{Connected component of id}_X \text{ in } \mathcal{H}^u(X, d)$

(1) A.V. Černavskiǐ:

$M = \text{the interior of a compact manifold } N$

the metric $d = \text{the restriction of some metric on } N$

$\implies \mathcal{H}^u(M, d) : \text{locally contractible}$

(2) K. Mine, K. Sakai, T. Yagasaki and A. Yamashita

$\mathcal{H}^u(\mathbb{R})_b \approx \ell_\infty$
§.2. Strategy

(1) Choose reasonable class of metric spaces.

Riemannian covering spaces over compact Riemannian manifolds

Metric covering spaces over compact metric manifolds

(2) Local Property

(i) Local deformation lemma for uniform embeddings

\[ \uparrow \quad \text{R. D. Edwards – R. C. Kirby} \]

Local deformation lemma for top. embeddings of compact subspaces

(ii) Local contractibility of groups of uniform homeomorphisms

(3) Global Property

Similarity transformations

Euclidean space \( \mathbb{R}^n \) \( \Rightarrow \) Global Homotopical Property

\( \rightarrow \) Euclidean ends \( \mathbb{R}^n - B(0, r) \) \( \rightarrow \) bi-Lipschitz Euclidean ends

\( \rightarrow \) General Metric spaces with bi-Lipschitz Euclidean ends
§.3. Choose reasonable class of metric spaces

Model example:

Riemannian covering spaces over compact Riemannian manifolds

\[ \pi : (X, d) \to (Y, \varrho) : \text{a covering projection between metric spaces} \]

**Def.** \( \pi : (X, d) \to (Y, \varrho) : \text{a metric covering projection} \)

\[ \iff \quad (\sharp)_1 \ \exists \ U : \text{an open cover of } Y \text{ s.t. } \forall \ U \in \mathcal{U} \]
\[ \pi^{-1}(U) = \bigcup_i U_i : \text{the disjoint union of open subsets of } X \]
\[ \text{each } U_i \text{ is mapped isometrically onto } U \text{ by } \pi \]

\[ (\sharp)_2 \ \forall \ y \in Y \quad \pi^{-1}(y) : \text{uniformly discrete in } X \]

\[ (\sharp)_3 \ \varrho(\pi(x), \pi(x')) \leq d(x, x') \quad (x, x' \in X) \]

\o\ A \subset X : \varepsilon\text{-discrete} \iff d(x, y) \geq \varepsilon \ (\forall \ x, y \in A, x \neq y) \]
Basic deformation theorem by R. D. Edwards – R. C. Kirby

for embeddings of a compact subset in a top manifold

\( M : \) a top \( n \)-manifold possibly with boundary

\( X, Y \subset M \)

\( \mathcal{E}_*(X, M; Y) = \) Space of proper embeddings \( f : X \to M \) s.t.

\[ f = \text{id} \text{ on } X \cap Y \] (Compact - Open topology)

Thm. \( C \subset M : \) a compact subset \( U : \) a neighborhood of \( C \) in \( M \)

\( K : \) a compact neighborhood of \( C \) in \( U \)

\( D \subset E \subset M : \) closed subsets s.t. \( D \subset \text{Int}_M E \)

\[ \implies \exists \ U : \text{a neighborhood of } i_U \text{ in } \mathcal{E}_*(U, M; E) \]

\( \varphi : U \times [0, 1] \longrightarrow \mathcal{E}_*(U, M; D) : \) a homotopy s.t.

(i) \( \forall f \in U \) \hspace{1cm} (a) \( \varphi_0(f) = f \), \hspace{1cm} (b) \( \varphi_1(f)|_C = i_C \)

\[ (c) \varphi_t(f) = f \text{ on } U - K \hspace{1cm} (t \in [0, 1]) \]

(ii) \( \varphi_t(i_U) = i_U \hspace{1cm} (t \in [0, 1]). \)
§.4. Local deformation lemma for uniform embeddings

$(M, d)$: Top manifold possibly with boundary  
$d$: a metric on $M$

$X, C \subset M$

$\mathcal{E}_u^u(X, M; C)$: Space of uniform proper embeddings

$$f : (X, d|_X) \rightarrow (M, d) \quad \text{s.t.} \quad f = \text{id} \quad \text{on} \quad X \cap C$$

- the uniform topology induced from the sup-metric
\( \pi : (M, d) \rightarrow (N, \varrho) \): a metric covering projection

\( N \): a compact top \( n \)-manifold possibly with boundary

**Thm 1.1.**

\( X \subset M \): a closed subset

\( W' \subset W \) are uniform neighborhoods of \( X \) in \((M, d)\)

\( Z \subset Y \): closed subsets of \( M \) s.t. \( Y \) is a uniform neighborhood of \( Z \).

\[ \exists \mathcal{W} : \text{a neighborhood of } \iota_W \text{ in } \mathcal{E}^u_*(W, M; Y) \]

\[ \exists \varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_*(W, M; Z) : \text{a homotopy s.t.} \]

\[ (1) \ \forall h \in \mathcal{W} \quad (i) \ \varphi_0(h) = h \quad (ii) \ \varphi_1(h) = \text{id on } X \]

\[ (iii) \ \varphi_t(h) = h \text{ on } W - W' \ (t \in [0, 1]) \]

\[ (2) \ \varphi_t(\iota_W) = \iota_W \ (t \in [0, 1]). \]

**Cor.** \( \mathcal{H}^u(M, d) \): locally contractible
Ideas of Proof of Thm 1.1.

General Case $\implies$ Product covering case

Open covering of $N$

$M = \bigcup_i M_i \supset X = \pi^{-1}(Y) = \bigcup_i X_i$

$\pi \downarrow \quad \downarrow$

$N \supset Y$

$E^u(X, M) \ni f \downarrow i_X$

Construct $\varphi_t(f) : f \simeq i_X$

\[ f \downarrow i_X \quad \varphi_t(f) \]

\[ \psi_t(g) : g \simeq i_Y \]
\( f : \) a uniform embedding
\[
\begin{array}{c}
\uparrow \\
\end{array}
\]
\( f_i = f|_{X_i} : X_i \to M_i \)
equi-uniform embeddings
\[
\begin{array}{c}
\uparrow \\
\end{array}
\]
\( \bar{f}_i : Y_i \to N : \) equi-uniform embeddings
\[
\begin{array}{c}
\uparrow \\
\end{array}
\]
\( cl\{\bar{f}_i\}_i : \) compact
\[
\begin{array}{c}
\Rightarrow \\
\end{array}
\]
(\textbf{Arzela-Ascoli theorem})

\( \varphi_i(f) = \bigcup_i \varphi_i(f_i) : \) a uniform embedding
\[
\begin{array}{c}
\uparrow \\
\end{array}
\]
\( \varphi_i(f_i) : X_i \to M_i \)
equi-uniform embeddings
\[
\begin{array}{c}
\uparrow \\
\end{array}
\]
\( \{\psi_i(\bar{f}_i)\}_i : \) equi-uniform embeddings
\[
\begin{array}{c}
\uparrow \\
\end{array}
\]
\( \psi_i(cl\{\bar{f}_i\}_i) : \) compact
§.5. Global Homotopical Property of $\mathcal{H}^u(X, d)$

Standard example:

$\mathbb{R}^n$: \textbf{$n$-dim Euclidean space} with the standard Euclidean metric

Similarity transformations:

Local deformation $\implies$ Global deformation
(Local = close to id) (Global = far from id)

Euclidean ends $\mathbb{R}^n_r := \mathbb{R}^n - O(r)$ ($r > 0$)

$\downarrow$ bi-Lipschitz equivalence

bi-Lipschitz Euclidean ends

\textbf{Metric spaces with bi-Lipschitz Euclidean ends}

$(X, d)$: a metric space

**Def.** a bi-Lipschitz $n$-dim Euclidean end of $(X, d) =$

$L \subset X$: a closed subset s.t.

$\exists \theta : (\mathbb{R}_1^n, \partial \mathbb{R}_1^n) \approx ((L, \text{Fr}_X L), d|_L)$: a bi-Lipschitz homeo. of pairs

$d(X - L, L_r) \to \infty$ as $r \to \infty$ \quad $L_r = \theta(\mathbb{R}_r^n)$ ($r \geq 1$)
Thm 1.2.

\((X, d)\) : a metric space

\(L(1), \cdots, L(m)\) : bi-Lipschitz Euclidean ends of \((X, d)\) mutually disjoint

\(L_r = L(1)_r \cup \cdots \cup L(m)_r \ (r \geq 1)\)

\(\implies \exists \ \text{SDR} \ \varphi_t \ \text{of} \ \mathcal{H}^u(X)_b \ \text{onto} \ \mathcal{H}^u_{L_3}(X)_b \ \text{s.t.} \)

\[ \varphi_t(h) = h \ \text{on} \ h^{-1}(X - L_2) - L_2 \ \text{for any} \ (h, t) \in \mathcal{H}^u(X)_b \times [0, 1] \]

SDR

\[ \mathcal{H}^u(X)_b \supset \mathcal{H}^u_{L_3}(X)_b \]

\[ \cup \]

\[ \mathcal{H}^u(X)_0 \supset \mathcal{H}^u_{L_3}(X)_0 \]

SDR
Examples

\[ \mathcal{H}^u(\mathbb{R}^n)_b \supset \mathcal{H}_{\mathbb{R}^3}(\mathbb{R}^n) \approx \mathcal{H}_\partial(B(3)) \simeq \ast \]

SDR

Remark

Alexander trick \[\implies\] a contraction of \( \mathcal{H}_\partial(B(1)) \)

\[ \exists \mathcal{H}^u(\mathbb{R}^n)_b \longrightarrow \mathcal{H}_\partial(B(1)) : \text{an injective map} \]

defined by

(i) take a shrinking homeomorphism \( \chi : \mathbb{R}^n \approx O(1) \)

(ii) conjugate by \( \chi \) and extend by id on \( \partial B(1) \)

The induced contraction of \( \mathcal{H}^u(\mathbb{R}^n)_b \) is not continuous.
(2) \( M = S^{n-1} \times \mathbb{R} \) with two bi-Lipschitz Euclidean ends

\[ \Rightarrow \mathcal{H}^u(M)_0 \supset \mathcal{H}_{L3}(M)_0 \approx \mathcal{H}_\partial(S^{n-1} \times [-1,1])_0 \]

SDR

(3) \( n = 2 \)

\[ M = N - C \]

\( N \): a compact connected 2-manifold with boundary

\( C = C_1 \cup \cdots \cup C_m \): some boundary circles of \( N \) \( (m \geq 1) \)

\( d \): a metric on \( M \) s.t. each ends of \( M \) is a bi-Lipschitz Euclidean ends

\[ \Rightarrow \mathcal{H}^u(M)_0 \supset \mathcal{H}_{L3}(M)_0 \approx \mathcal{H}_C(N)_0 \simeq * \]

SDR
End of Talk

Thank you very much!