Groups of uniform homeomorphisms of covering spaces
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Motivation
The uniform topology is one of basic topologies on function spaces. We study local and global topological properties of spaces of uniform embeddings and groups of uniform homeomorphisms in metric manifolds endowed with the uniform topology. Since the notions of uniform continuity and uniform topology depend on the choice of metrics, we are concerned with dependence of those properties on the behavior of metrics in neighborhoods of ends of manifolds.

Basic Definitions and Notations
A map \( f : (X,d) \to (Y,g) \) between metric spaces is said to be uniformly continuous if \( \forall \varepsilon > 0 \exists \delta > 0 \) such that if \( x,x' \in X \) and \( d(x,x') < \delta \) then \( g(f(x),f(x')) < \varepsilon \). The map \( f \) is called a uniform homeomorphism if \( f \) is bijective and both \( f \) and \( f^{-1} \) are uniformly continuous. An uniform embedding is a uniform homeomorphism onto its image. For subsets \( A,B \) of a metric space \((X,d)\), we say that \( B \) is a uniform neighborhood of \( A \) in \( (X,d) \) and write \( A \subseteq_{u} B \) if \( B \) contains the \( \varepsilon \)-neighborhood \( O_{\varepsilon}(A) \) of \( A \) in \( X \) for some \( \varepsilon > 0 \).

A subset \( A \) of \( X \) is said to be uniformly discrete if there exists an \( \varepsilon > 0 \) such that \( d(x,y) \geq \varepsilon \) for any distinct points \( x,y \in A \).

For each \( x \) in \( X \) the fiber \( \pi^{-1}(x) \) is uniformly discrete.

The newt definition is motivated by the Edwards-Kirby local deformation theorem for embeddings of compact spaces [1].

Definition 1. A subset \( A \) of \( M \) we say that "\( A \) has the local deformation property for uniform embeddings in \( (M,d)\)" and write \( A : (LDM) \) if the following holds:

\( (*) \) for any tuple \( (X,W,W',Z,Y) \) of subsets of \( M \)

\[ X \subseteq A \subseteq_{u} X, X \subseteq_{u} W \subseteq C \subseteq_{u} Z \subseteq_{u} Y \subseteq M \]

there exists a neighborhood \( W \subseteq C \subseteq_{u} W \subseteq M \) in \( E_{n}(W, M, Y) \) and a homotopy \( \varphi : W \times [0,1] \to E_{n}(W, M, Y) \) such that for each \( f \in W \)

\((i) \) \( \varphi(f) = f \) on \( X \),

\((ii) \) \( \varphi(f) \) is \( \varphi \)-on \( X \),

\((iii) \) \( \varphi(f) = f \) on \( W' - W \) and \( \varphi(f)(W) = f(W) \) \( (t \in [0,1]) \),

\((iv) \) \( f \) is \( \varphi \)-on \( W \cap M \), then \( f \) is \( \varphi \)-on \( W \cap M \).

We omit the subscript \( M \) in the symbol \((LDM)\) when \( A = M \).

The Edwards-Kirby deformation theorem \([1]\) can be restated in the next form.

Edwards-Kirby deformation theorem.

\( K \subset M \) : Relatively compact \( \Rightarrow K : (LDM) \)

Basic Properties of \((LDM)\).

\( (1) \) \( \text{Invariance under Uniform Homeo's} \)

\( (2) \) \( \text{Restriction} \)

\( (3) \) \( \text{Additivity} \)

\( (4) \) \( \text{Relatively compact subsets} \)

\( (5) \) \( \text{Neighborhoods of Ends} \)

Example 1. Metric covering spaces over compact manifolds.

The following notion is a natural metric version of Riemannian covering projections.

Definition 2. A map \( \pi : (X,d) \to (Y,g) \) is called a metric covering projection if it satisfies the following conditions:

\( (1) \) There exists an open cover \( U \) of \( Y \) such that for each \( U \in U \) the inverse \( \pi^{-1}(U) \) is the disjoint union of open subsets of \( X \) each of which is mapped isometrically onto \( U \) by \( \pi \).

\( (2) \) For each \( y \in Y \) the fiber \( \pi^{-1}(y) \) is uniformly discrete in \( X \).

\( (3) \) \( \pi : (M,d) \to (N,g) \) : a metric covering projection \( \Rightarrow (M,d) : (LD) \)

This follows from Theorem 1 and Additivity of \((LD)\).

Theorem 1. \( \pi : (M,d) \to (N,g) \) a metric covering projection \( \Rightarrow (M,d) : (LD) \)

References