Schrödinger operators on the metric lattice

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Abstract. We consider Schrödinger operators with decreasing potentials on the metric lattice. The metric lattice is the simplest metric graph $\mathbb{M}^d = (\mathbb{Z}^d, \mathcal{E})$, where $\mathbb{Z}^d$ is the vertex set and the edge set $\mathcal{E}$ is given by

$$\mathcal{E} = \big\{(m, m + e_j), \quad \forall\, m \in \mathbb{Z}^d, j = 1, .., d\big\}, \tag{0.1}$$

and $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$ is the standard orthonormal basis in $\mathbb{R}^d$. Each edge $\mathbf{e} \in \mathcal{E}$ of $\mathbb{M}^d$ will be identified with the segment $[0, 1]$. This identification introduces a local coordinate $t \in [0, 1]$ along each edge. For each function y on $\mathbb{M}^d$ we define a function $y_{\mathbf{e}} = y\big|_{\mathbf{e}}$, $\mathbf{e} \in \mathcal{E}$. We identify each function $y_{\mathbf{e}}$ on $\mathbf{e}$ with a function on $[0, 1]$ by using the local coordinate $t \in [0, 1]$. Let $L^2(\mathbb{M}^d)$ be the Hilbert space of all function $y = (y_{\mathbf{e}})_{\mathbf{e}\in\mathcal{E}}$, where each $y_{\mathbf{e}} \in L^2(\mathbf{e}) = L^2(0, 1)$, equipped with the norm $\|y\|_{L^2(\mathbb{M}^d)}$, where $L^p(\mathbb{M}^d), p \geqslant 1$ is given by

$$\|y\|^p_{L^p(\mathbb{M}^d)} = \sum_{\mathbf{e}\in\mathcal{E}} \|y_{\mathbf{e}}\|^p_{L^p(\mathbf{e})} < \infty.$$

We define the metric Laplacian Δ_M on $y = (y_{\mathbf{e}})_{\mathbf{e}\in\mathcal{E}} \in L^2(\mathbb{M}^d)$ by

$$(\Delta_M y)_{\mathbf{e}} = -y''_{\mathbf{e}}, \qquad plus \quad so-called\; Kirchhoff\; conditions.$$

The Laplacian $H_0 = \Delta_M \geqslant 0$ and has the spectrum

$$\sigma(H_0) = \sigma_{ac}(H_0) \cup \sigma_{fb}(H_0), \qquad \sigma_{ac}(H_0) = [0, \infty), \quad \sigma_{fb}(H_0) = \{\pi^2 n^2, n \in \mathbb{N}\}.$$

where $\sigma_{fb}(H_0)$ is the set of all flat bands (eigenvalues with infinite multiplicity). We consider Schrödinger operators $H = H_0 + Q$ on $\mathbb{M}^d$, where the real potential $Q \in L^1(\mathbb{M}^d)$. We have the following results:

Let $v = |Q|^{\frac{1}{2}}$ and $d \geqslant 3$. Then the operator-valued function

$$v(H_0 - \cdot)^{-1} P_{ac}(H_0) v : \mathbb{C} \setminus [0, \infty) \to \mathcal{B}$$

is analytic and Hölder continuous up to the boundary, where $\mathcal{B}$ is the class of bounded operators. Furthermore, the wave operators

$$W_\pm = s - \lim e^{itH} e^{-itH_0} P_{ac}(H_0) \qquad as \qquad t \to \pm\infty \tag{0.2}$$

exist and are complete, i.e., $\mathscr{H}_{\rm ac}(H) = \operatorname{Ran} W_\pm$.

Furthermore, we describe the eigenvalues of the Schrödinger operators $H = H_0 + Q$.

If the potential Q is uniformly decaying, then we obtain the Mourre estimates for the free metric Laplacian and describe more exactly the eigenvalues of the Schrödinger operators H.

It is the joint result with Jacob Schach Moller and Morten Grud Rasmussen, Denmark.